Tank Blowdown Math

Dean Wheeler
Brigham Young University
February 4, 2019

Introduction

This document provides a mathematical model for computing the rate of expelling gas through a small orifice or nozzle attached to a tank. Furthermore, two models are described for how fast the tank will depressurize. Related material on compressible flow can be found in fluid mechanics and thermodynamics textbooks and web pages.

Fig. 1 shows the tank and associated nozzle. The narrowest diameter of the flow path in the orifice or nozzle is known as the throat region. The tank and throat regions are described with their own sets of equations.

Provided the tank is large and the throat is small, it will take many seconds to empty the tank and gas velocities in the main part of the tank will be much smaller than the speed of sound. This means that gas pressure, temperature, and density in the tank will be spatially uniform, though they will be changing in time. Thus, we describe the tank using a transient mass balance. One can compare this to a model in heat transfer known as *lumped capacitance*.

In the nozzle region however, gas velocity is large and there are large spatial variations in the gas properties. In addition, there is relatively little gas contained in the nozzle region. Thus, the nozzle adjusts its flowrate to rapidly match current conditions in the tank, making it seem as if the nozzle is operating at steady state. This approximation for the nozzle is known as quasi-steady state.

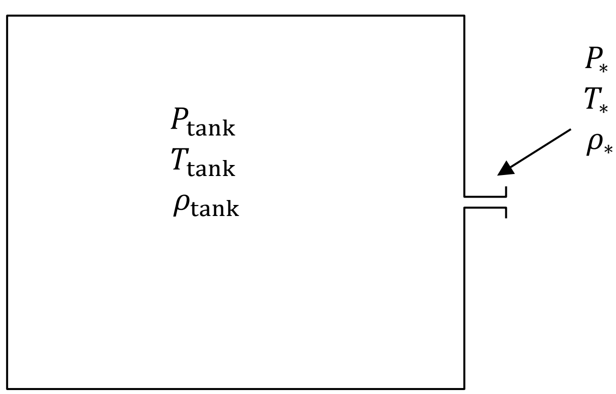

Fig. 1. Schematic of a tank with nozzle or orifice, allowing gas to exit. Indicated are variables that pertain to two key regions. Every variable depends on time.

Equation of State

The P, T, and ρ variables in Fig. 1 denote absolute pressure, absolute temperature, and density in the tank or the narrowest part of the nozzle or throat (subscript *), respectively. Note that if tank pressure is given experimentally as a gauge quantity, it must be converted to absolute to be used in the equations below.

The first relationship between gas variables is given by an equation of state. The ideal gas law is a fairly accurate representation for air when pressure is less than around 10 atm or 150 psia. It states that

$$PV = nRT \tag{1}$$

where V is the volume of the gas, n is the number of moles, and R is the universal gas constant (8.31446 J/mol-K). With the introduction of the molecular weight M (effectively 0.028964 kg/mol for air), and the substitution that density is mass over volume $\rho = nM/V$, the ideal gas law is changed to

$$\rho = \frac{PM}{RT} \tag{2}$$

This equation could be applied separately to the tank variables or to the throat variables.

Temperature and Pressure During Expansion

The second important relationship comes from figuring out what happens when gas in the tank or nozzle expands. When a gas expands, its internal energy is used to perform work on the surroundings, and the gas therefore tends to cool off. If the gas expands slowly, there is time for it to absorb heat from its warmer surroundings and the expansion is essentially isothermal, meaning the temperature stays at its initial value or that of the surroundings.

On the other hand, if a gas expands quickly its temperature will drop dramatically. This is called adiabatic expansion, where adiabatic means no noticeable heat transfer from the surroundings (i.e. the walls of the tank). In adiabatic expansion, the pressure drops more rapidly than it would for isothermal (slow) expansion. Adiabatic expansion could happen inside the tank if it is emptying rapidly, but this depends on the relative sizes of the tank and nozzle. On the other hand, adiabatic expansion certainly occurs when a gas moves from the tank through the nozzle region. In other words, here the gas is moving quickly and therefore expanding quickly.

The relationships for pressure and temperature for reversible adiabatic expansion are

$$P = P_0 \left(\frac{\rho}{\rho_0}\right)^{\gamma} \tag{3a}$$

$$T = T_0 \left(\frac{\rho}{\rho_0}\right)^{\gamma-1} \tag{4a}$$

where the subscript 0 indicates the initial state of the gas before the expansion started. This means if we know how the density is changing from an initial state to some later state, we can compute P and T as well. In the case of the nozzle, we apply the above equations as the gas travels between the tank and the throat. In that case, they become

$$P_* = P_{\text{tank}} \left(\frac{\rho_*}{\rho_{\text{tank}}}\right)^{\gamma} \tag{3b}$$

$$T_* = T_{\text{tank}} \left(\frac{\rho_*}{\rho_{\text{tank}}}\right)^{\gamma-1} \tag{4b}$$

The parameter γ is the dimensionless ratio of specific heats ($\gamma = c_p/c_v$), and by statistical theory of gases $\gamma = 7/5 = 1.4$ for low-temperature diatomic molecules. Air is mostly composed of diatomic molecules like O_2 and N_2 and so that value is used here.

Choked Flow

Next, we need to determine the gas density in the nozzle when the tank is at a specified condition. Recall that that the nozzle is treated as if it instantaneously responds to whatever state the tank is in. A more-full discussion of the nozzle flow equations can be found in other sources like textbooks that cover ideal compressible flow in nozzles.

Choked flow means that the flow is exactly at the speed of sound in the throat region. A higher speed cannot be achieved in the throat, regardless of upstream or downstream conditions. Thus, choked flow acts to limit how much gas flow can pass through a given size orifice. This is the reason why pressure relief valves on tanks must be properly sized to accommodate sufficient flow.

Choked flow happens for a large pressure drop across the nozzle or orifice, specifically if the upstream tank pressure meets the following condition relative to atmospheric pressure downstream from the nozzle:

$$P_{\text{tank}} \geq P_{\text{atm}} \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}} \tag{5}$$

Eq. 5 is the origin of the rule of thumb or approximation that choked flow occurs for upstream pressure that is more than twice the value of downstream pressure (absolute). If the tank pressure drops below this limit, the speed of gas in the throat is subsonic, and less gas will flow than in the choked-flow regime. The solution to subsonic flow in the nozzle is complicated and is less important to know because it is at the end of the tank's discharge when pressure is low, and so will be neglected here.

The solution to choked flow in the throat region follows a simple relationship:

$$\frac{\rho_*}{\rho_{\text{tank}}} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \tag{6}$$

This can be substituted into Eqs. 3b and 4b to determine pressure and temperature in the throat in terms of tank conditions.

For choked flow the throat velocity is exactly the speed of sound, which is what makes it easier to analyze. For ideal gases, speed of sound c is determined solely by temperature. Thus, we can relate throat velocity to throat temperature, and in turn to tank temperature:

$$v_* = c_* = \left(\gamma \frac{RT_*}{M}\right)^{\frac{1}{2}} = \left(\frac{2\gamma}{\gamma+1}\frac{RT_{\text{tank}}}{M}\right)^{\frac{1}{2}} \tag{7}$$

Mass Flow Rate

Now we can determine the mass flow rate $\dot{m}$ through the nozzle or orifice. This comes from the following standard relationship, applied at the throat, because that is where conditions are known:

$$\dot{m} = C_d A_* \rho_* v_* \tag{8}$$

where A_* is the throat cross-sectional area given by

$$A_* = \frac{\pi}{4} d_*^2 \tag{9}$$

and where d_* is throat diameter.

Dimensionless parameter C_d in Eq. 8 is the *discharge coefficient*, accounting for friction between fluid and walls and a phenomenon known as *vena contracta*. In essence, C_d is needed in Eq. 8 because the effective area for fluid at speed v_* is somewhat smaller than actual throat area. C_d would be 1 for a perfect (thermodynamically reversible) nozzle: in practice for a smoothly tapering nozzle it might be as high as 0.98, while for a sharp-edged orifice it might be as low as 0.60. Anything that causes separation of flow from the nozzle wall or increases frictional contact will decrease C_d.

Making the appropriate substitutions into Eq. 8 leads to an equation for mass flow in terms of readily determined quantities:

$$\dot{m} = C_d A_* \, \rho_{\text{tank}} \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \left(\frac{2\gamma}{\gamma+1}\frac{RT_{\text{tank}}}{M}\right)^{\frac{1}{2}}$$

$$= C_d A_* \, \rho_{\text{tank}} \left(\gamma \frac{RT_{\text{tank}}}{M}\right)^{\frac{1}{2}} \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{10}$$

Frequently in industrial situations, mass flow rates are expressed instead as volumetric flow rates corresponding to a gas at a standard temperature and pressure (even though the gas is not actually at that temperature and pressure). For instance, a mass flow meter used for gases may express mass flow as standard liters per minute (SLM) or standard cubic feet per minute (SCFM). In other words, even though $\dot{m}$ is the key value being measured, it is expressed as

$$\dot{V}_{\text{std}} = \frac{\dot{m}}{\rho_{\text{std}}} \tag{11}$$

which requires knowing what ρ_{std} value is programmed by the manufacturer into the flow meter. This can be determined from the ideal gas law, given specified P_{std} and T_{std} values. As an example, if the manufacturer takes $T_{\text{std}} = 70°\text{F} = 294.26\text{ K}$ and $P_{\text{std}} = 1\text{ atm} = 14.7\text{ psia}$, then $\rho_{\text{std}} = 1.200\text{ kg/m}^3$.

Combining Eqs. 10 and 11 and the ideal gas law leads to

$$\dot{V}_{\text{std}} = C_d A_* c_{\text{std}} \left(\frac{P_{\text{tank}}}{P_{\text{std}}}\right) \left(\frac{T_{\text{std}}}{T_{\text{tank}}}\right)^{\frac{1}{2}} \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{12}$$

where c_{std} is the speed of sound at the standard temperature:

$$c_{\text{std}} = \left(\gamma \frac{RT_{\text{std}}}{M}\right)^{\frac{1}{2}} \tag{13}$$

Makers of orifices (and valves) may provide an experimentally determined size parameter known as *flow coefficient* C_v. For gases this dimensionless parameter can be converted to $C_d A_*$ by

$$C_d A_* = C_v \cdot 16.2\text{ mm}^2 \tag{14}$$

The key design principles resulting from the above analysis are, provided tank pressure is large enough to generate choked flow, that (1) the mass flow rate of a gas through an orifice is proportional to throat area and tank pressure and (2) flow rate does not depend on downstream pressure.

Two Models of Tank Blowdown

Eq. 10 gives the rate of mass loss from a tank at a given gas density and temperature. To determine how long it will take to depressurize the tank, we must do a transient mass balance on the tank. The ordinary differential equation for this is

$$\frac{dm}{dt} = -\dot{m} \tag{15}$$

where $\dot{m}$ comes from Eq. 10 and m is mass of gas in the tank. This is turn is

$$m = \rho_{\text{tank}} V_{\text{tank}} \tag{16}$$

where V_{tank} is the fixed tank volume. With these substitutions we get for the governing equation

$$V_{\text{tank}} \frac{d\rho_{\text{tank}}}{dt} = -C_d A_*\, \rho_{\text{tank}} \left(\gamma \frac{RT_{\text{tank}}}{M}\right)^{\frac{1}{2}} \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{17}$$

To make things more manageable, let us create a *discharge time constant* called τ:

$$\tau = \frac{V_{\text{tank}}}{C_d A_* c_0} \left(\frac{\gamma + 1}{2} \right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{18}$$

where c_0 is the speed of sound at initial temperature T_0 (i.e. at the beginning of blowdown):

$$c_0 = \left(\gamma \frac{RT_0}{M} \right)^{\frac{1}{2}} \tag{19}$$

With this new time constant Eq. 17 becomes

$$\frac{d\rho_{\text{tank}}}{dt} = -\frac{\rho_{\text{tank}}}{\tau} \left(\frac{T_{\text{tank}}}{T_0} \right)^{\frac{1}{2}} \tag{20}$$

The last thing to do before solving this equation is figure out what to do with T_{tank}. We have two options:

Isothermal tank. Assume gas temperature in the tank does not change in time, based on blowdown taking a long time so that heat can be readily absorbed from the walls. Thus, $T_{\text{tank}} = T_0$. This leads to Eq. 20 becoming

$$\frac{d\rho_{\text{tank}}}{dt} = -\frac{\rho_{\text{tank}}}{\tau} \tag{21}$$

which can be separated and integrated to give solution

$$\rho_{\text{tank}} = \rho_0 \exp\left(-\frac{t}{\tau} \right) \tag{22}$$

where ρ_0 is initial density in the tank. We then convert densities to pressures using the ideal gas equation.

$$P_{\text{tank}} = P_0 \exp\left(-\frac{t}{\tau} \right) \tag{23}$$

This equation tells us how tank pressure varies with time, for an isothermal tank and choked exit flow.

Adiabatic tank. Assume the gas cools as it expands in the tank, due to no heat transfer from the walls, based on blowdown taking a short time. Thus, T_{tank} is given by Eq. 4a. This leads to Eq. 20 becoming

$$\frac{d\rho_{\text{tank}}}{dt} = -\frac{\rho_{\text{tank}}}{\tau} \left(\frac{\rho_{\text{tank}}}{\rho_0} \right)^{\frac{\gamma-1}{2}} \tag{24}$$

which can be separated and integrated to give solution

$$\rho_{\text{tank}} = \rho_0 \left[1 + \left(\frac{\gamma - 1}{2} \right) \frac{t}{\tau} \right]^{\frac{2}{1-\gamma}} \tag{25}$$

We then convert densities to pressures using Eq. 3a for adiabatic expansion.

$$P_{\text{tank}} = P_0 \left[1 + \left(\frac{\gamma - 1}{2}\right)\frac{t}{\tau}\right]^{\frac{2\gamma}{1-\gamma}} \tag{26}$$

This equation tells us how tank pressure varies with time, for an adiabatic tank and choked exit flow. The tank temperature can likewise be predicted from Eq. 4a

$$T_{\text{tank}} = T_0 \left[1 + \left(\frac{\gamma - 1}{2}\right)\frac{t}{\tau}\right]^{-2} \tag{27}$$

Comparison. The isothermal and adiabatic models of tank blowdown can be considered two extremes, with the correct answer (i.e. with the true amount of heat transfer) lying somewhere in between them. Fig. 2 shows an example of the respective blowdown curves (Eqs. 23 and 26). As noted previously, adiabatic tank conditions lead to more rapid pressure loss than do isothermal conditions.

The curves predict that the tank will have lost 80% of its original pressure at a time in the range $1.3\tau < t < 1.6\tau$. This shows the value of evaluating the variable τ to get an approximation of the time it takes to depressurize the tank.

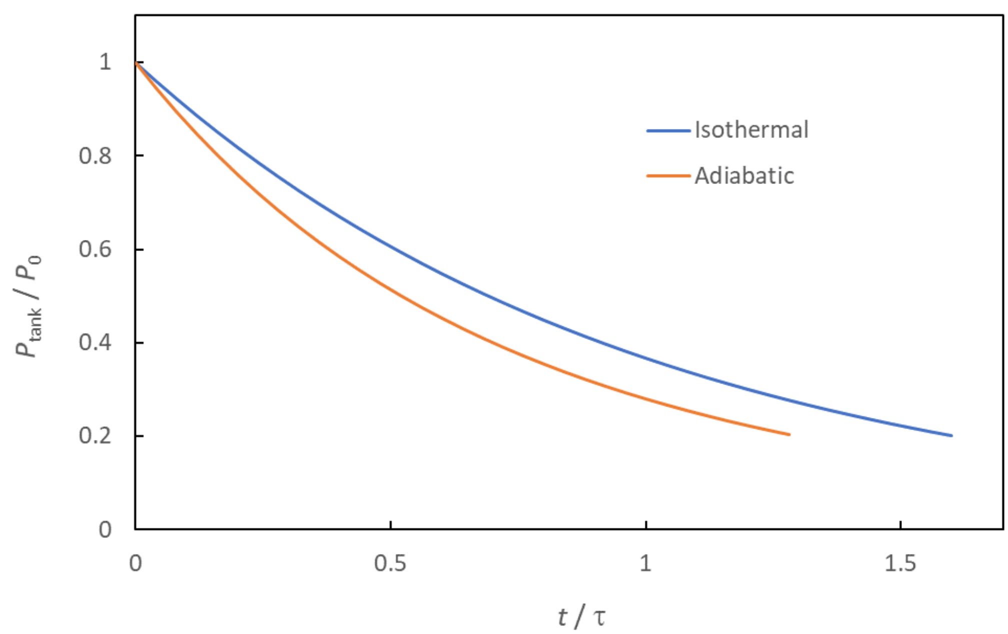

Fig. 2. Comparison of isothermal and adiabatic blowdown curves.